Conics – The Ellipse – Center at Origin

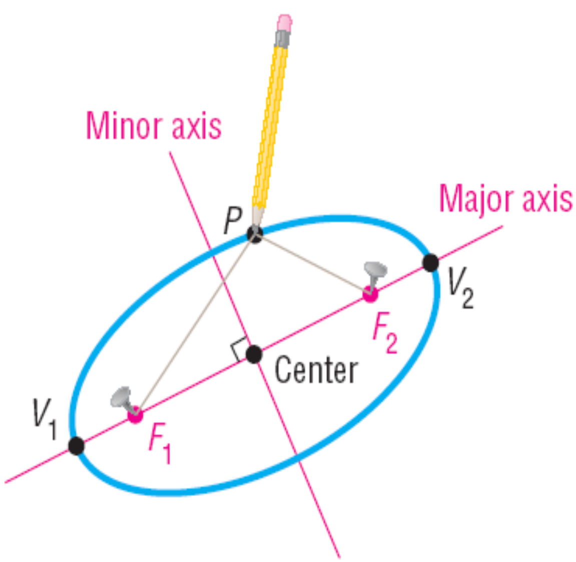
This section covers the following topics:

- **Define terms: Ellipse, Foci and major and minor axes, center**
- **Theorems of equations of ellipses**
- **Analyze the equation of an ellipse and find equation of ellipse**

<u>Define Parabola, Directrix and Focus</u>

Definition – Ellipse

- An ellipse is the collection of all points in the plane, the sum of whose distances form two fixed point, called the foci, is a constant.



- The distance from F_1 to P and P to F_2 is the same at any point on the ellipse; that is, let P_i be any point (x, y) on the ellipse. Then $d(F_1, P_i) = d(F_2, P_i)$
- The line containing the foci is called a major axis and the line through the center and perpendicular to the major axis is the minor axis.
- The midpoint of the line segment joining the foci is the center of the ellipse.
- The two points of intersection of the ellipse and the major axis are the vertices V_1 and V_2. The distance from one vertex to the other is the length of the major axis.

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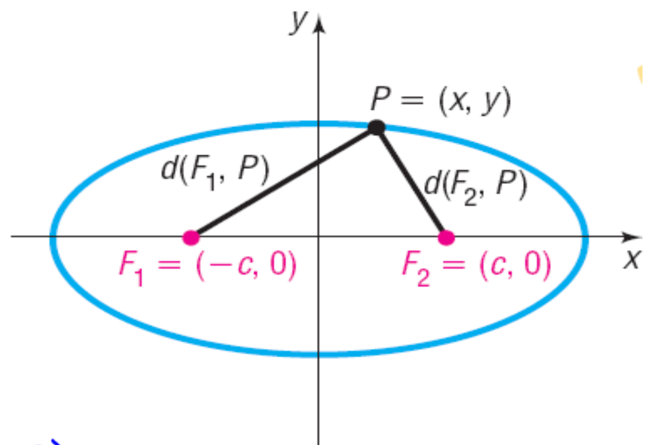

$$d(F_1,P)+d(F_2,P)=2a$$

$$\sqrt{[(x-(-c)]^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$$

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

$$\left(\sqrt{(x+c)^2+y^2}\right)^2=\left(2a-\sqrt{(x-c)^2+y^2}\right)^2$$

$$(x+c)^2+y^2=(2a)^2-2(2a)\sqrt{(x-c)^2+y^2}+\left(\sqrt{(x-c)^2+y^2}\right)^2$$

$$(x+c)^2+\cancel{y^2}=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+\cancel{y^2}$$

$$\cancel{x^2}+2cx+\cancel{c^2}=4a^2-4a\sqrt{(x-c)^2+y^2}+\cancel{x^2}-2cx+\cancel{c^2}$$

$$\frac{1}{4}(4cx-4a^2)=\left(-4a\sqrt{(x-c)^2+y^2}\right)\frac{1}{4}$$

$$(cx-a^2)^2=\left(-a\sqrt{(x-c)^2+y^2}\right)^2$$

$$c^2x^2-2cxa^2+a^4=a^2\left((x-c)^2+y^2\right)$$

$$c^2x^2-2ca^2x+a^4=a^2(x^2-2cx+c^2+y^2)$$

$$c^2x^2-\cancel{2ca^2x}+a^4=a^2x^2-\cancel{2ca^2x}+a^2c^2+a^2y^2$$

$$a^4-a^2c^2=a^2x^2-c^2x^2+a^2y^2$$

$$a^2(a^2-c^2)=(a^2-c^2)x^2+a^2y^2$$

$$a>c$$

$$d(F_1,P)+d(F_2,P)>d(F_1,F_2)$$

$$2a>2c$$

$$a > c$$

and $a > c > 0$

So $a^2 > c^2$

and $a^2 - c^2 > 0$

Let $b^2 = a^2 - c^2$, $b > 0$,

then $a > b$ and we have

$$\frac{a^2 b^2}{a^2 b^2} = \left(b^2 x^2 + a^2 y^2\right)\frac{1}{a^2 b^2}$$

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

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Theorems of equations of ellipses

Theorem: Equation of an ellipse – Vertex $(0,0)$, Major axis along x-axis

The equation of an ellipse with center $(0,0)$, foci $(-c,0)$ and $(c,0)$ and vertices at $(-a,0)$ and $(a,0)$ is:

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a > b > 0$ and $b^2=a^2-c^2$

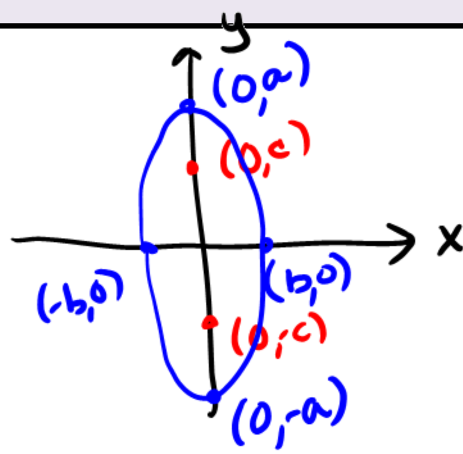
The major axis is the x-axis.

Theorem: Equation of an ellipse – Vertex $(0,0)$, Major axis along y-axis

The equation of an ellipse with center $(0,0)$, foci $(0,-c)$ and $(0,c)$ and vertices at $(0,-a)$ and $(0,a)$ is:

$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, where $a > b > 0$ and $b^2=a^2-c^2$

The major axis is the y-axis.



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Analyze the equation of an ellipse and find equation of ellipse

Example 1: Find the vertices and foci of the ellipse. Then graph the ellipse.

$$\frac{x^2}{36} + \frac{y^2}{4} = 1 \rightarrow \frac{x^2}{(6)^2} + \frac{y^2}{(2)^2} = 1$$

so $a = 6$, $b = 2$

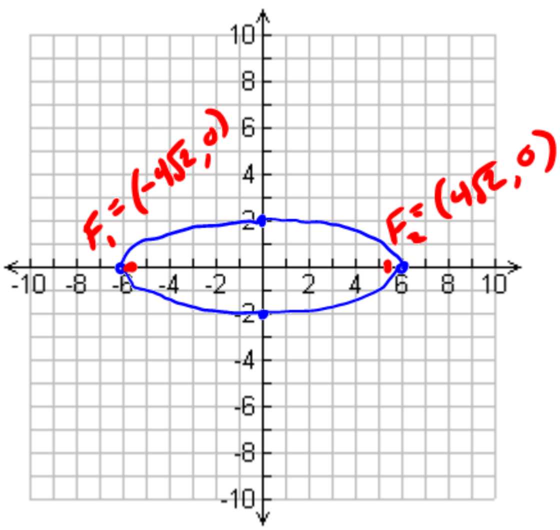
$a^2 - c^2 = b^2$ $\quad c > 0$

$36 - c^2 = 4$

$\sqrt{32} = \sqrt{c^2}$

$4\sqrt{2} = c$

$c \approx 5.7$



Example 2: : Find the vertices and foci of the ellipse. Then graph the ellipse.

$$\frac{x^2}{9} + \frac{y^2}{25} = 1 \rightarrow \frac{(x-0)^2}{(3)^2} + \frac{(y-0)^2}{(5)^2} = 1$$

$a = 5$

$b = 3$

$c^2 = a^2 - b^2$

$c^2 = 25 - 9$

$c^2 = 16$

$c = 4$

$F_1 = (0, 0-4)$

$F_1 = (0, -4)$

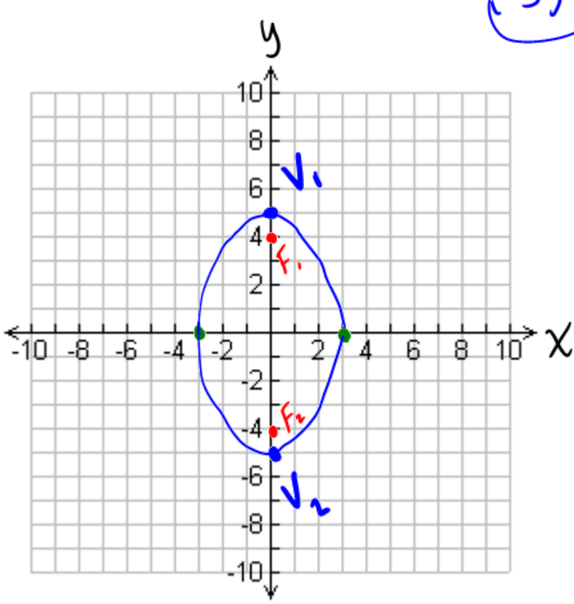
$F_2 = (0, 0+4)$

$F_2 = (0, 4)$

$V_1 = (0, 0+5)$

$V_1 = (0, 5)$

$V_2 = (0, -5)$



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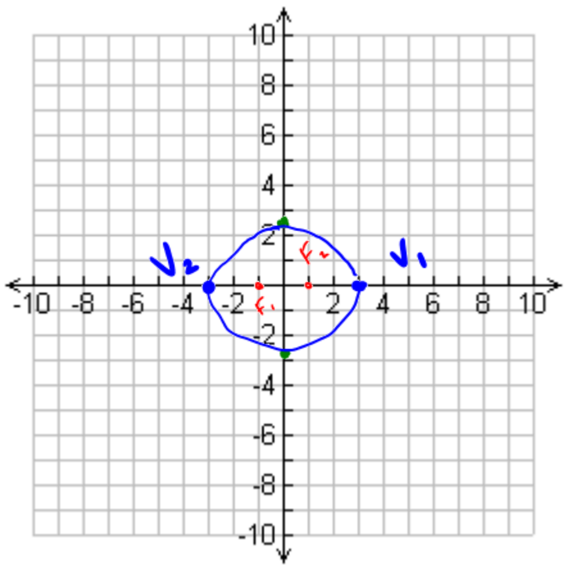
Example 3: Find the equation for the ellipse with center $(0,0)$; focus $(-1,0)$ and vertex $(3,0)$



$$\frac{x^2}{\boxed{a}^2}+\frac{y^2}{\boxed{b}^2}=1$$

$$\frac{x^2}{(3)^2}+\frac{y^2}{\boxed{2\sqrt{2}}^2}=1$$

$$\frac{x^2}{9}+\frac{y^2}{8}=1$$

$F_1=(-1,0)$

Shift of 1 unit to the left of the center.

$F_2=(0+1,\ 0)$

$F_2=(1,0)\rightarrow c=1$

$V_1=(3,0)$

$V_2=(0-3,0)$

$V_2=(-3,0)$

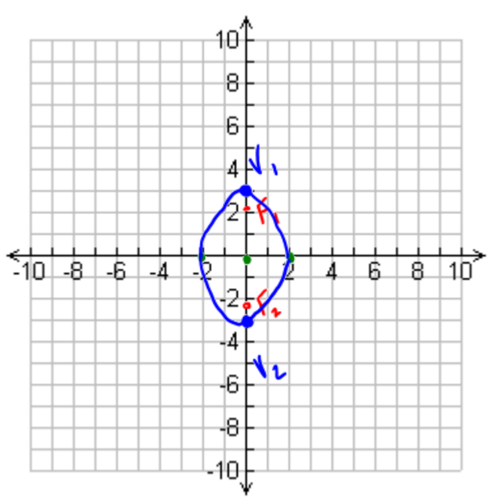
$c^2=a^2-b^2$

$1^2=3^2-b^2 \rightarrow b^2=8$

$b=2\sqrt{2}$

$b\approx 2.8$

Example 4: Analyze the equation of the parabola: $4y^2+9x^2=36$.



$$\frac{4y^2+9x^2}{36}=\frac{36}{36}$$

$$\frac{y^2}{9}+\frac{x^2}{4}=1$$

$$\frac{x^2}{(2)^2}+\frac{y^2}{(3)^2}=1$$

$a=3$ major axis is vertical

$b=2$

$c^2=a^2-b^2$

$c^2=9-4$

$c^2=5$

$c=\sqrt{5}\approx 2.2$

$V_1=(0,3)$

$V_2=(0,-3)$

center at $(0,0)$

$F_1=(0,\sqrt{5})\approx(0,2.2)$

$F_2=(0,-\sqrt{5})\approx(0,-2.2)$